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Analytical Geometry of two dimensions.

Q.1. Show that the equation $12x^2+2y^2-10xy+11x-5y+2=0$ can be reduced to a homogeneous equation of the second degree by transferring the origin to a properly chosen point.

Soln: Let the origin be transferred to the point (h,k). If the point (x,y) becomes (x_1,y_1) referred to new axes, then

$$x=x_1+h \text{ and } y=y_1+k$$

The given equation becomes

$$12(x_1+h)^2+2(y_1+k)^2-10(x_1+h)(y_1+k)+11(x_1+h)-5(y_1+k)+2=0$$

$$\Rightarrow 12x_1^2+2y_1^2-10x_1y_1+x_1(24h-10k+11)+y_1(4k-10h-5)$$
$$+(12h^2+2k^2-10hk+11h-5k+2)=0$$

This equation will be reduced to a homogeneous equation of second degree, if

$$24h-10k+11=0,\quad 4k-10h-5=0$$

and $12h^2+2k^2-10hk+11h-5k+2=0$

$$\therefore \frac{h}{-50+44}=\frac{k}{110-120}=\frac{1}{-96+100}$$

$$\Rightarrow \frac{h}{-6}=\frac{k}{-10}=\frac{1}{4} \Rightarrow h=-\frac{6}{4} \text{ and } k=-\frac{10}{4}$$

$$\Rightarrow h=-\frac{3}{2} \text{ and } k=-\frac{5}{2}$$

Now, $12h^2+2k^2-10hk+11h-5k+2$

$$=12\times\frac{9}{4}+2\times\frac{25}{4}-10\times\frac{3}{2}\times\frac{5}{2}-11\times\frac{3}{2}+5\times\frac{5}{2}+2$$

$$=27+\frac{25}{2}-\frac{75}{2}-\frac{33}{2}+\frac{25}{2}+2=54-\frac{75}{2}-\frac{33}{2}=\frac{108-108}{2}=\frac{0}{2}=0$$

Hence, if the origin is transferred to $\left(-\frac{3}{2},-\frac{5}{2}\right)$, the given eqn reduces to $12x^2+2y^2-10xy=0$, which is a homogeneous eqn of the second degree.

Q.2. By transferring the origin to the point $(2,3)$ and turning the axes through an angle $\frac{\pi}{4}$, find the transformed form of the equation $3x^2+2xy+3y^2-18x-22y+50=0$

Soln: Shifting the origin to the point $(2,3)$, the transformed eqn becomes

$$3(x+2)^2+2(x+2)(y+3)+3(y+3)^2-18(x+2)-22(y+3)+50=0$$

$$\Rightarrow \cancel{3x^2+2xy+3y^2=1}$$

$$\Rightarrow 3(x^2+4x+4)+2(xy+3x+2y+6)+3(y^2+6y+9)-18x-36-22y-66+50=0$$

$$\Rightarrow 3x^2+\cancel{12x}+12+2xy+\cancel{6x}+\cancel{4y}+12+3y^2+\cancel{18y}+27-\cancel{18x}-36-\cancel{22y}-66+50=0$$

$$\Rightarrow 3x^2+2xy+3y^2+101-102=0$$

$$\Rightarrow 3x^2+2xy+3y^2-1=0.$$

$$\Rightarrow 3x^2+2xy+3y^2=1$$

Let now the axes be rotated through an angle $\frac{\pi}{4}$, then the transformed eqn becomes

$$3\left(x\cos\tfrac{\pi}{4}-y\sin\tfrac{\pi}{4}\right)^2+2\left(x\cos\tfrac{\pi}{4}-y\sin\tfrac{\pi}{4}\right)\left(x\sin\tfrac{\pi}{4}+y\cos\tfrac{\pi}{4}\right)+3\left(x\sin\tfrac{\pi}{4}+y\cos\tfrac{\pi}{4}\right)^2=1$$

$$\Rightarrow 3\left(x\times\tfrac{1}{\sqrt2}-y\times\tfrac{1}{\sqrt2}\right)^2+2\left(x\times\tfrac{1}{\sqrt2}-y\times\tfrac{1}{\sqrt2}\right)\left(x\times\tfrac{1}{\sqrt2}+y\times\tfrac{1}{\sqrt2}\right)+3\left(x\times\tfrac{1}{\sqrt2}+y\times\tfrac{1}{\sqrt2}\right)^2=1$$

$$\Rightarrow 3\left(\frac{x-y}{\sqrt2}\right)^2+2\left(\frac{x-y}{\sqrt2}\right)\left(\frac{x+y}{\sqrt2}\right)+3\left(\frac{x+y}{\sqrt2}\right)^2=1$$

$$\Rightarrow 3\frac{(x-y)^2}{2}+\frac{2(x^2-y^2)}{2}+3\frac{(x+y)^2}{2}=1$$

$$\Rightarrow \frac{3}{2}\left[(x-y)^2+(x+y)^2\right]+x^2-y^2=1$$

$$\Rightarrow \frac{3}{2}\left[2x^2+2y^2\right]+x^2-y^2=1$$

$$\Rightarrow \frac{3}{\cancel{2}}\times\cancel{2}(x^2+y^2)+x^2-y^2=1$$

$$\Rightarrow 3x^2+3y^2+x^2-y^2=1$$

$$\Rightarrow 4x^2+2y^2=1.$$

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Differential Calculus

Q. 1 State and prove the Taylor's theorem on the expansion of $f(x+h)$.

Soln. Statement: → Let the function $f(x+h)$ can be expanded in a series of powers of h, then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\underline{|2}} f''(x) + \frac{h^3}{\underline{|3}} f'''(x) + \cdots + \frac{h^n}{\underline{|n}} f^n(x) + \cdots \text{ to } \infty$$

Proof: It is given that $f(x+h)$ can be expanded in a series of integral powers of h.

$$\therefore \text{ Let } f(x+h) = A_0 + A_1 \frac{h}{\underline{|1}} + A_2 \frac{h^2}{\underline{|2}} + A_3 \frac{h^3}{\underline{|3}} + A_4 \frac{h^4}{\underline{|4}} + \cdots \text{ to } \infty \quad \text{--- (I)}$$

Where $A_0, A_1, A_2, A_3, A_4, \cdots$ are functions of x only and are to be determined.

We shall let us suppose that the series (I) is convergent and that the successive differential co-efficients $f'(x)$, $f''(x)$, $f'''(x)$, $\cdots$, $f^n(x)$ all exist.

$$\text{Now, } \frac{d}{dh} f(x+h) = \frac{d}{d(x+h)} f(x+h) \cdot \frac{d(x+h)}{dh} = f'(x+h)(0+1)$$
$$= f'(x+h)$$

As x and h are independent quantities and so x may be considered constant in differentiating with respect to h.

Similarly $\frac{d^2 f(x+h)}{dh^2} = f''(x+h)$, $\frac{d^3}{dh^3} f(x+h) = f'''(x+h)$ and so on.

Differentiating both sides of (I) with respect to h successively and the above differentiations, We get

$$f'(x+h) = 0 + A_1 \cdot 1 + A_2 h + A_3 \frac{h^2}{\underline{|2}} + A_4 \frac{h^3}{\underline{|3}} + \cdots \quad \text{II.}$$

$$f''(x+h) = A_2 + A_3 h + A_4 \frac{h^2}{\underline{|2}} + \cdots \text{———} \quad \text{III.}$$

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Rest part of Q.1 B.Sc. P-II, paper-III 13-04-2020

$\therefore f'''(x+h) = A_3 + A_4 h + \cdots \cdots$ IV

$\cdots \cdots \cdots$

Putting $h=0$ in (I), II, III, IV, $\cdots$, we get

$A_0 = f(x)$, $A_1 = f'(x)$, $A_2 = f''(x)$, $A_3 = f'''(x)$, $\cdots$

Substituting these values in I, we get

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) + \frac{h^3}{\lfloor 3}f'''(x) + \cdots + \frac{h^n}{\lfloor n}f^n(x) + \cdots \text{to } \infty$$

Q.2. Expand $\cos x$ in powers of $x-\frac{1}{4}\pi$.

Soln. Here, $\cos x = \cos\left(x-\frac{\pi}{4}+\frac{\pi}{4}\right) = \cos\left(y+\frac{\pi}{4}\right)$

$= f\left(y+\frac{\pi}{4}\right)$, where $y = x-\frac{\pi}{4}$

By Taylor's series, we have

$$f\left(y+\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) + yf'\left(\frac{\pi}{4}\right) + \frac{y^2}{\lfloor 2}f''\left(\frac{\pi}{4}\right) + \cdots \text{to } \infty \quad \text{— I}$$

Now, $f\left(y+\frac{\pi}{4}\right) = \cos\left(y+\frac{\pi}{4}\right) \Rightarrow f(z) = \cos z$ $\therefore f^n(z) = \cos\left(\frac{n\pi}{2}+z\right)$

$\Rightarrow f^n\left(\frac{\pi}{4}\right) = \cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$

$\therefore f'\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2}+\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$f''\left(\frac{\pi}{4}\right) = \cos\left(\pi+\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$\cdots \cdots \cdots$

Also, $f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore$ from I, we get

$$\cos\left(y+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \left(x-\frac{\pi}{4}\right)\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{\lfloor 2}\left(-\frac{1}{\sqrt{2}}\right) + \cdots \text{to } \infty$$

$$\Rightarrow \cos x = \frac{1}{\sqrt{2}}\sqrt{2}\left[1-\left(x-\frac{\pi}{4}\right) - \frac{\left(x-\frac{\pi}{4}\right)^2}{\lfloor 2} + \cdots \text{to } \infty\right]$$

Which is the required expansion.